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Transcendental Function Meditations

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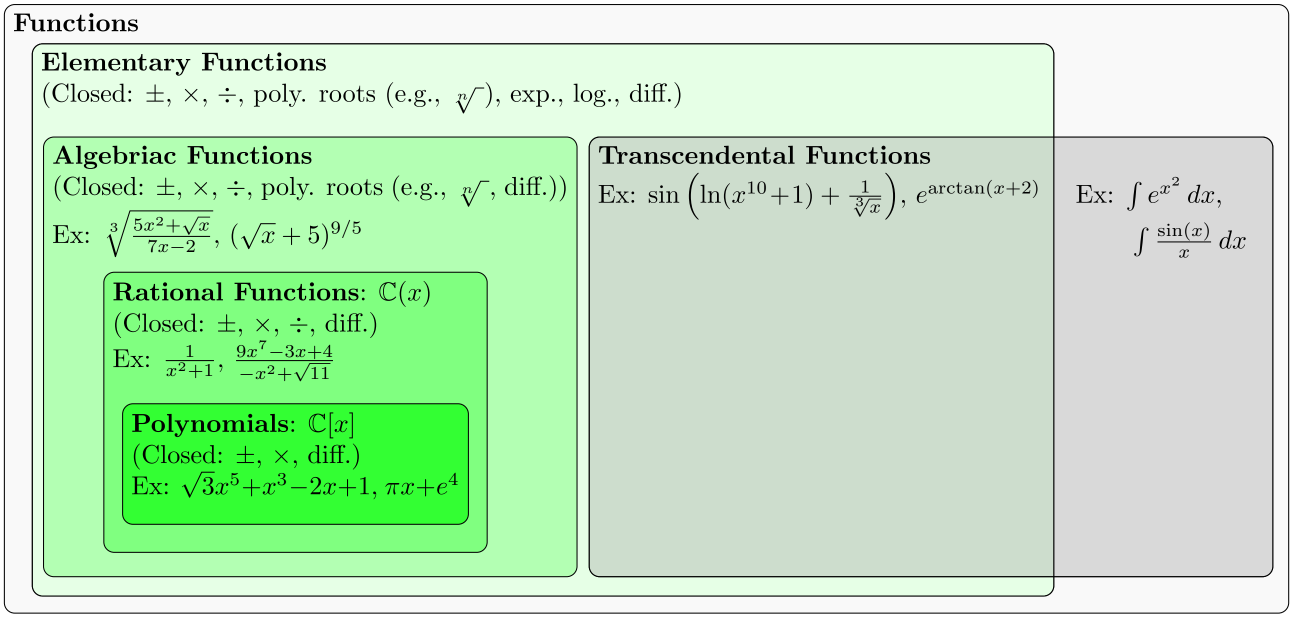
Abstract

What is a transcendental function? Whether they realize it or not, most students are quite familiar with these functions. In this paper, we first seek to unravel and explain the terms "algebraic" and "transcendental" in the context of numbers and then in the context of functions. We then prove some basic results establishing that exponential, logarithmic, trigonometric, and inverse trigonometric functions are transcendental. We also briefly discuss the nature of an "elementary function.". The paper concludes with a series of reader investigations exploring these concepts in a computer algebra system such as Maple.

1 Why Ponder the Transcendental?

There are many types of functions, including polynomial, rational, algebraic, transcendental, and elementary functions (many of which seem far from *elementary*). We deal with these kinds of functions every day in ordinary algebra and calculus courses. In terms of symbolic manipulation, there is a sizable gulf between algebraic (e.g., $\sqrt{x}$) and transcendental (e.g., e^x) functions. Given the ubiquity of such functions, it seems wise to understand what makes them different and how they arise. Paralleling operational closures leading to the development of various familiar number systems, we seek to elucidate such collections of functions and indicate how they arise when seeking various forms of closure.

We begin this discussion with a figure that sums up the culmination of our investigation. Here we show nested collections function types with some indications of the kinds of closures these systems possess. This is intended to motivate interest among readers who may wish to visually organize polynomial, rational, transcendental, elementary, and others into subset and superset relationships based on notions of closure. We hope this diagrammatic representation will encourage others to investigate a hierarchical relationship among these functions. Notably, this document's contents will explain much of the symbolism in the diagram.



2 A Numerical Preamble

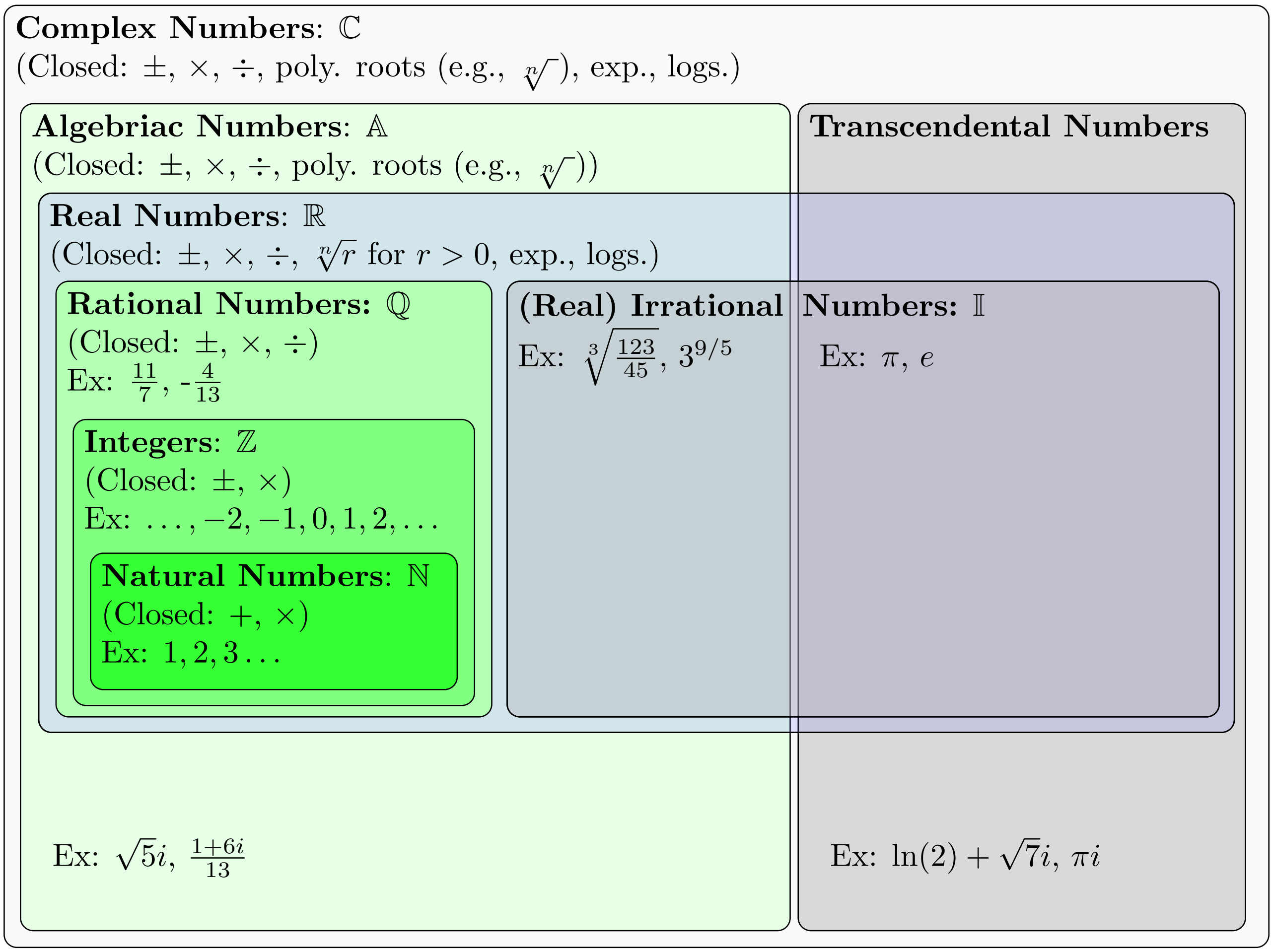
Before diving into the world of functions, we seek to gain a concrete sense of *algebraic* versus *transcendental* in the context of familiar number systems. For a more detailed exploration of various number systems, we recommend [2].

2.1 Number Systems and Operational Closure

The story of number systems can be viewed through the lens of various closures. We understand that the natural numbers (i.e., positive integers), $\mathbb{N} = \{1, 2, 3, 4, \dots\}$, are closed under addition and multiplication and include the multiplicative identity. If we wish to include the additive identity, we must consider the whole numbers (i.e., non-negative integers), $\mathbb{W} = \{0, 1, 2, 3, 4, \dots\}$. Suppose we wish to include closure for subtraction. We must then step up to the integers, $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, which include additive inverses.[1] Demanding closure under division (as always, except by zero) leads us to the rational numbers, $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}$, which include multiplicative inverses.

[1]For those with some abstract algebra background, in this paper, we will only consider rings with characteristic zero (i.e., repeatedly adding a non-zero element to itself will never yield zero). Every such ring (with multiplicative identity) contains an isomorphic copy of the integers (its *prime subring*) and thus gives us a natural starting point.

While this might seem like the end of the road, the rational numbers ($\mathbb{Q}$) are not closed under root taking (e.g., $\sqrt{2} \notin \mathbb{Q}$) as well as many other operations. This leads to the need for additional number systems such as the real numbers ($\mathbb{R}$), the real irrational numbers ($\mathbb{I}$), the algebraic numbers ($\mathbb{A}$), and complex numbers ($\mathbb{C}$). A nesting of some number systems is provided in the following figure.



2.2 Caressing (*Lightly Touching*) Algebraic Structures

We recall names for systems with various closure properties. Without giving all the formal details (see [4]), a *commutative ring* (with unity) is a non-empty collection equipped with operations of addition, subtraction, and multiplication. A good example of such is the integers ($\mathbb{Z}$). If we also want division (except by zero), we have a system called a *field*. The rational numbers ($\mathbb{Q}$), real numbers ($\mathbb{R}$), and complex numbers ($\mathbb{C}$) should be familiar examples of fields.

The real numbers include irrational numbers like $\sqrt{2}$ and $\sqrt[3]{5}$ but also wilder numbers such as $\pi \approx 3.14159265$ and $e \approx 2.7182818$. The real numbers are not only closed under root taking (for non-negative real numbers) but also closed under a kind of limiting. In fact, one can characterize the real numbers as being the only ordered field closed under taking greatest lower bounds and least upper bounds (i.e., it the the only complete ordered field up to order preserving isomorphism [5]).

2.3 Wildly Irrational Numbers

Two considerations arise from the above discussion. First, what about roots of negative numbers? If we demand root taking for all numbers, we must introduce i where $i^2 = -1$ (i.e., the imaginary root).[2] Gauss, in his dissertation, was able to prove that the complex numbers are *algebraically closed* [8]. This means that $\mathbb{C}$ is not only closed under addition, subtraction, multiplication, division, and root taking but also under extracting roots of any polynomial relation.

Second, what exactly makes π wilder than $\sqrt{2}$? This concern is essentially the main consideration of this paper. The difference between π and $\sqrt{2}$ is that π transcends algebraic (i.e., polynomial) relationships among rational numbers, whereas $\sqrt{2}$ satisfies such a relation (e.g., $(\sqrt{2})^2 = 2$). In particular, we define the notion of an algebraic number:

Definition 2.1 *A complex number $z \in \mathbb{C}$ is* **algebraic** *if there are rational numbers $a_n, \ldots, a_0 \in \mathbb{Q}$, not all zero, such that $a_n z^n + \cdots + a_1 z + a_0 = 0$. In other words, a number is algebraic if it is the root of a non-zero polynomial with rational coefficients. In contrast, any number that fails to be algebraic is called* **transcendental**.

After briefly considering the above definition, it should be apparent why showing a number to be algebraic is often pretty easy. For example, $\sqrt[3]{5}$ is algebraic because it is a root of $x^3 - 5$ (end of proof). On the other hand, showing a number is transcendental is almost always very challenging – challenging but not impossible. In 1873, Charles Hermite showed that Euler's number e is transcendental, and in 1882, Ferdinand von Lindemann proved that π is transcendental [3]. To explore a large collection of transcendental numbers with patterned decimal expansion, see [1]. It should be noted that algebraic numbers are not always easily recognized. For example, $\cos(\frac{\pi}{9})$ is algebraic since it is a root of the cubic polynomial $8x^3 - 6x - 1$, but this is not exactly obvious.

2.4 Planting Some Ideas

This section comes with the following disclaimer: *Here, we provide an overview of ideas that will be addressed in more detail in future sections. While we acknowledge the immediate topical jump in complexity, we ask the reader to persevere and trust that explanations are coming.*

[2] We eschew the notation $\sqrt{-1}$ for the imaginary root i. Why? For example, $\sqrt{4}$ denotes *the* non-negative number that squares to be 4. Thus $\sqrt{4} = 2$. If we just carelessly state that $\sqrt{4}$ is *the* number that squares to 4, we are left with ambiguity for both ± 2 accomplish this. Now if one writes $\sqrt{-1}$, then this should be the *positive* number that squares to be -1. However, there is no way to linearly order the complex numbers in a way compatible with their arithmetic, so neither $\pm i$ can be considered *positive*. Another reason to avoid this notation is that it misleads one into making fallacious arguments such as: $-1 = (\sqrt{-1})^2 = \sqrt{(-1)^2} = \sqrt{1} = 1$.

One must be careful about the general notions of algebraic and transcendental. They are defined relative to some base system. When someone refers to algebraic or transcendental numbers without mentioning a base system, they typically mean they are working over the rational numbers ($\mathbb{Q}$).

More generally, if $\mathbb{K}$ is a field[3] containing another field $\mathbb{F}$ (i.e., $\mathbb{K}$ is an *extension field* of $\mathbb{F}$, or equivalently $\mathbb{F}$ is a *subfield* of $\mathbb{K}$), then we say an element $\alpha \in \mathbb{K}$ is *algebraic* over $\mathbb{F}$ if it is a root of some nonzero polynomial with coefficients in $\mathbb{F}$: $c_n\alpha^n + \cdots + c_1\alpha + c_0 = 0$ for some $c_n, \ldots, c_1, c_0 \in F$ where not all c_i's are zero. Any element $\alpha \in \mathbb{K}$ which fails to be algebraic over $\mathbb{F}$ is said to be *transcendental* over $\mathbb{F}$. This means that while an algebraic element in $\mathbb{K}$ might not belong to $\mathbb{F}$, it is not *too far* away in terms of algebraic entanglements. In contrast, a transcendental element has no algebraic relationship (in terms of adding, subtracting, multiplying, dividing) to the base system!

We emphasize that *the base system matters*. Notice that while π is transcendental over $\mathbb{Q}$, it is algebraic over $\mathbb{R}$ since π is the root of $x - \pi$ (a polynomial whose coefficients 1 and $-\pi$ belong to $\mathbb{R}$). In fact, working over $\mathbb{R}$, every element of $\mathbb{C}$ is algebraic, whereas working over $\mathbb{Q}$, most of the elements of $\mathbb{C}$ are transcendental.[4] Trivially, if $c \in \mathbb{F}$, then c is the root of $x - c$ whose coefficients (i.e., 1 and $-c$) belong to $\mathbb{F}$. Thus, unsurprisingly, every element of $\mathbb{F}$ is algebraic over $\mathbb{F}$ itself.

3 A Brief Account of *Some* Mathematical Field Theory

We note that algebraic numbers possess some closure properties. Consider some field $\mathbb{K}$ extending another field $\mathbb{F}$ with $\alpha, \beta \in \mathbb{K}$. If α and β are algebraic over $\mathbb{F}$, then so are $\alpha + \beta$, $\alpha - \beta$, $\alpha \cdot \beta$, and α/β (if $\beta \neq 0$). In fact, roots of algebraic numbers are still algebraic. Loosely speaking, doing algebraic stuff to algebraic elements yields algebraic elements (working over some fixed base field $\mathbb{F}$). For example, since $\sqrt{2}$ and $\sqrt[3]{5}$ are algebraic (over $\mathbb{Q}$), we have $\sqrt{2} - \sqrt[3]{5}$ and $\sqrt[9]{\sqrt[3]{5} + 123\sqrt{2}}$ are algebraic as well.[5]

Let us briefly explain why the statements above hold and simultaneously introduce some useful notation and concepts. In linear algebra, we learn that a vector space is a collection of elements equipped with a (vector) addition and scalar multiplication. Introductory courses generally restrict scalars to the fields of real ($\mathbb{R}$) or sometimes complex ($\mathbb{C}$) numbers. However, one can work more generally with any field $\mathbb{F}$ as their field of scalars. Notably, a vector space V working over a field of scalars $\mathbb{F}$ always has a basis, and any two bases have the same number of elements. This common size of a basis is called the *dimension* of the vector space and is often denoted $\dim_{\mathbb{F}}(V)$ (or leaving out the $\mathbb{F}$ subscript if our field of scalars is understood). For example, $\dim_{\mathbb{F}}(\mathbb{F}^3) = 3$ if $\mathbb{F}^3$ denotes ordered triples with entries in $\mathbb{F}$. For example, we could use $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ (a set of 3 elements) as a basis.

Given a field $\mathbb{K}$ extending a field $\mathbb{F}$, we can view $\mathbb{K}$ as a vector space with scalars in $\mathbb{F}$ (we certainly

[3]Recall that a field is an abstract system closed under addition, subtraction, multiplication, and division.

[4]Given $z = a + bi \in \mathbb{C}$ where $a, b \in \mathbb{R}$, we have $(x - z)(x - \overline{z}) = (x - a - bi)(x - a + bi) = x^2 - 2ax + (a^2 + b^2)$ is a real polynomial where z is a root. Thus z is algebraic over $\mathbb{R}$. On the other hand, one can show that there are only *countably* many algebraic numbers, whereas there are *uncountably* many real and complex numbers. Thus "most" of these numbers must be transcendental.

[5]Good luck guessing a rational polynomial with such roots! There are algorithms that do this. Computer algebra systems such as Maple can find such polynomials for us if we would like.

can add elements of $\mathbb{K}$ and multiply elements of $\mathbb{K}$ by elements of $\mathbb{F}$). In mathematical field theory, one defines $[\mathbb{K}:\mathbb{F}] = \dim_{\mathbb{F}}(\mathbb{K})$ to be the *degree* of the extension of $\mathbb{K}$ working over $\mathbb{F}$. In other words, the degree of an extension is the dimension of that field viewed as a vector space over the base field. For example, every complex number can be uniquely expressed as $a+bi$ where $a,b \in \mathbb{R}$. This means $\{1,i\}$ is a basis for $\mathbb{C}$ viewed as a real vector space and so $[\mathbb{C}:\mathbb{R}] = 2$. On the other hand, $[\mathbb{C}:\mathbb{C}] = 1$ (it is always true that $[\mathbb{F}:\mathbb{F}] = 1$) and $[\mathbb{C}:\mathbb{Q}]$ is infinite.[6]

We need some notations for extending a number system to a larger one. First, given a commutative ring (with multiplicative identity) R, the notation $R[t]$ denotes the smallest ring accommodating all of R and the "new" element t. A general element of $R[t]$ looks like $c_nt^n+\cdots+c_1t+c_0$ where $c_n,\ldots,c_1,c_0 \in R$ (i.e., a general element is a polynomial in t with coefficients drawn from R). Next, $R(t)$ denotes the smallest ring accommodating all of R and the new element t and its multiplicative inverse $1/t$. A general element in $R(t)$ looks like $f(t)/g(t)$ where $f(t)$ and $g(t)$ are elements of $R[t]$ (with $g(t) \neq 0$). In other words, these are rational functions in t with coefficients in R. If R is a field, then $R(t)$ is the smallest field containing all of R as well as t. For example, since even powers of $\sqrt{2}$ are of the form 2^k and odd powers of $\sqrt{2}$ are of the form $2^k\sqrt{2}$, it turns out that elements of $\mathbb{Q}[\sqrt{2}]$ are of the form $a+b\sqrt{2}$ for some $a,b \in \mathbb{Q}$. Moreover, by the conjugate trick we learned in our youth,

$$\frac{a+b\sqrt{2}}{c+d\sqrt{2}} = \frac{a+b\sqrt{2}}{c+d\sqrt{2}} \cdot \frac{c-d\sqrt{2}}{c-d\sqrt{2}} = \frac{ac-2bd}{c^2-2d^2} + \frac{-ad+bc}{c^2-2d^2}\sqrt{2},$$

ratios of elements in $\mathbb{Q}[\sqrt{2}]$ are still elements in $\mathbb{Q}[\sqrt{2}]$. Thus $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}[\sqrt{2}]$. Similarly, $\mathbb{C} = \mathbb{R}[i] = \mathbb{R}(i)$. When x is an indeterminate, $R[x]$ simply denotes the ring of polynomials with coefficients in R and $R(x)$ denotes rational functions with coefficients in R. For example, $\dfrac{\pi x^2 - \sqrt{3}}{-x^3+5x+\sqrt[5]{11}}$ belongs to the field of real rational functions, $\mathbb{R}(x)$.

3.1 Algebraic and Transcendental Elements

Now, we return to algebraic versus transcendental elements. Using our new notation: Given a field $\mathbb{K}$ extending a field $\mathbb{F}$, recall that $\alpha \in \mathbb{K}$ is algebraic over $\mathbb{F}$ if $f(\alpha) = 0$ for some nonzero polynomial $f(x) \in \mathbb{F}[x]$. It is useful to collect all polynomials with root α together into a collection (called an *ideal*). Indeed, given any algebraic (over $\mathbb{F}$) element α, there is a unique monic (i.e., its leading coefficient is 1) polynomial $m(x) \in \mathbb{F}[x]$ of lowest degree such that $m(\alpha) = 0$. Such a polynomial is irreducible in $\mathbb{F}[x]$ (i.e., it does not factor in a non-trivial way in $\mathbb{F}[x]$), and if $f(x) \in \mathbb{F}[x]$ is such that $f(\alpha) = 0$, then $f(x)$ is a multiple of $m(x)$. We call $m(x)$ the *minimal polynomial* of α. For example, x^2-2 is the minimal polynomial of $\sqrt{2}$ working over $\mathbb{Q}$. Notice that $\sqrt{2}$ is also a root of x^4-4. So, $x^4-4 = (x^2+2)(x^2-2)$ is a multiple of x^2-2. Once again, the base field matters. The minimal polynomial of $\sqrt{2}$ working over $\mathbb{R}$ is just $x-\sqrt{2}$. Notice that $x^2-2 = (x+\sqrt{2})(x-\sqrt{2})$ factors over $\mathbb{R}$ but not over $\mathbb{Q}$.

When α is algebraic over $\mathbb{F}$, $\mathbb{F}[\alpha]$ is always a field (i.e., $\mathbb{F}[\alpha] = \mathbb{F}(\alpha)$) and $[\mathbb{F}[\alpha]:\mathbb{F}]$ is precisely the degree of α's minimal polynomial. For example, since every element of $\mathbb{Q}[\sqrt{2}]$ can be uniquely written

[6]Technically $[\mathbb{C}:\mathbb{Q}] = 2^{\aleph_0} = \mathfrak{c}$ since $\mathbb{C}$ requires a basis of continuum cardinality when working over the rationals.

as $a + b\sqrt{2}$ for some $a, b \in \mathbb{Q}$, we have that $\{1, \sqrt{2}\}$ is a basis for $\mathbb{Q}[\sqrt{2}]$ (as a vector space over $\mathbb{Q}$), so $[\mathbb{Q}[\sqrt{2}] : \mathbb{Q}] = \dim_{\mathbb{Q}}(\mathbb{Q}[\sqrt{2}]) = 2$. Alternatively, we see that $[\mathbb{Q}[\sqrt{2}] : \mathbb{Q}] = \deg(x^2 - 2) = 2$.

On the other hand, if α is transcendental over $\mathbb{F}$, our element α behaves like an indeterminate. In this case, $\mathbb{F}[\alpha]$ is always a proper subset of $\mathbb{F}(\alpha)$. Also, both $[\mathbb{F}[\alpha] : \mathbb{F}]$ and $[\mathbb{F}(\alpha) : \mathbb{F}]$ are infinite.[7] For example, $\mathbb{Q}[\pi]$ behaves just like the ring of polynomials with rational coefficients, $\mathbb{Q}[x]$. Putting this all together, α is algebraic over $\mathbb{F}$ if and only if $\mathbb{F}[\alpha] = \mathbb{F}(\alpha)$ if and only if $[\mathbb{F}[\alpha] : \mathbb{F}]$ is finite where the degree of this extension is precisely the degree of our algebraic element's minimal polynomial. On the other hand, α is transcendental over $\mathbb{F}$ if and only if $\mathbb{F}[\alpha] \subsetneq \mathbb{F}(\alpha)$ if and only if $[\mathbb{F}[\alpha] : \mathbb{F}]$ is infinite.

If the degree $[\mathbb{K} : \mathbb{F}]$ is finite, then we say $\mathbb{K}$ is a finite extension of $\mathbb{F}$. The above paragraph tells us that $\mathbb{F}[\alpha]$ is a finite extension of $\mathbb{F}$ exactly when α is algebraic over $\mathbb{F}$. One can easily show if $\mathbb{K}$ is a finite (degree) extension of $\mathbb{F}$, then all the elements of $\mathbb{K}$ are algebraic over $\mathbb{F}$. The converse is not true. It is possible to have an infinite (degree) extension containing only algebraic elements.

The *degree formula* is an important tool when working with these concepts. It states that given a field $\mathbb{L}$ extending a field $\mathbb{K}$ which in turn extends a field $\mathbb{F}$, we have

$$[\mathbb{L} : \mathbb{F}] = [\mathbb{L} : \mathbb{K}] \cdot [\mathbb{K} : \mathbb{F}].$$

Consequently, finite extensions of finite extensions are still finite extensions. This also implies that if α is algebraic over $\mathbb{K}$ and $\mathbb{K}$ is an algebraic extension of $\mathbb{F}$, then α is algebraic over $\mathbb{F}$. For example, $\mathbb{Q}[\sqrt[3]{5}]$ is algebraic over $\mathbb{Q}$ since $[\mathbb{Q}[\sqrt[3]{5}] : \mathbb{Q}] = \deg(x^3 - 5) = 3$ is finite. Notice that $\sqrt{1 + \sqrt[3]{5}}$ is a root of $x^2 - (1 + \sqrt[3]{5}) \in (\mathbb{Q}[\sqrt[3]{5}])[x]$. Thus $\sqrt{1 + \sqrt[3]{5}}$ is algebraic over $\mathbb{Q}[\sqrt[3]{5}]$ and thus also algebraic over $\mathbb{Q}$.

All algebraic elements in a field $\mathbb{K}$ (working over a base field $\mathbb{F}$) collected together form a field (a subfield of $\mathbb{K}$ and an extension field of $\mathbb{F}$). In other words, adding, subtracting, multiplying, and dividing (except by zero) algebraic elements yields an algebraic element. Therefore, if t is transcendental and $a \neq 0$ is algebraic, then $a \cdot t$ must be transcendental (otherwise $t = (a \cdot t) \cdot \frac{1}{a}$ is a product of algebraic elements and thus algebraic). Likewise, if t is transcendental and a is algebraic, then $t+a$ is transcendental (otherwise, $t = (t + a) + (-a)$ is the sum of algebraic elements and thus algebraic). This means that numbers like $\sqrt{2} + \pi$ and $5^{1/3} \cdot e$ are transcendental (over $\mathbb{Q}$).

4 Fields of Functions

Recall that when someone says a real or complex number is algebraic (respectively transcendental) without referencing a base field, we understand that they mean it is algebraic (respectively transcendental) over the rational numbers ($\mathbb{Q}$). When we move the realm of functions, we let our default base field be the field of rational functions with complex coefficients.

Definition 4.1 *We let x denote an indeterminate (essentially our real or complex variable) and then $\mathbb{C}(x) = \left\{ \frac{f(x)}{g(x)} \;\middle|\; f(x), g(x) \in \mathbb{C}[x] \text{ and } g(x) \neq 0 \right\}$ is the field of rational polynomials with complex coefficients. Given a complex function $h(x)$, we say $h(x)$ is* **algebraic** *(respectively* **transcendental***) if it is algebraic (respectively transcendental) over $\mathbb{C}(x)$.*

[7] More precisely, both $[\mathbb{F}[\alpha] : \mathbb{F}]$ and $[\mathbb{F}(\alpha) : \mathbb{F}]$ are $\aleph_0$ (countably infinite dimensional).

Why is $\mathbb{C}(x)$ a natural base field? To explain this, we first note why $\mathbb{Q}$ was a natural base field for numbers. We noted that if we want a system possessing the number 1 with closures under addition, subtraction, and multiplication, we must include all of the integers ($\mathbb{Z}$). If we also demand closure under division (except by zero), we must include all rational numbers ($\mathbb{Q}$). For functions, we wish to start off with all "numbers" (i.e., constant functions), and so $\mathbb{C}$ is a starting point. We should also include the so-called identity function $f(x) = x$. If we wish to accommodate all constants ($\mathbb{C}$), the identity function, and also require closures under addition, subtraction, and multiplication, then we must include all complex coefficient polynomials ($\mathbb{C}[x]$). If we also demand closure under division (except by zero), we are forced to include all complex coefficient rational functions ($\mathbb{C}(x)$).[8]

Since x is now playing the role of our variable, we will use a different symbol, Y, as our indeterminate when referring to polynomial relations. We restate our main definition: we have $h(x)$ is an algebraic (complex or real) function if there exists some non-zero polynomial $m(Y) \in (\mathbb{C}(x))[Y]$ such that $m(h(x)) = 0$. In other words, we have $\frac{f_n(x)}{g_n(x)}(h(x))^n + \cdots + \frac{f_1(x)}{g_1(x)}h(x) + \frac{f_0(x)}{g_0(x)} = 0$ for some complex polynomials $f_n(x), \ldots, f_1(x), f_0(x), g_n(x), \ldots, g_1(x), g_0(x) \in \mathbb{C}[x]$ where $g_i(x) \neq 0$ for all $i = 0, 1, \ldots, n$ and at least one $f_j(x) \neq 0$.

To make life easier we notice that we could clear the denominators in our above relation and get the equivalent definition: $h(x)$ is an algebraic function if there exists complex polynomials $c_n(x), \ldots, c_1(x)$, $c_0(x) \in \mathbb{C}[x]$ (not all zero) such that $c_n(x)(h(x))^n + \cdots + c_1(x)h(x) + c_0(x) = 0$. In other words, $h(x)$ is an algebraic function if and only if it is a root of some non-zero polynomial with complex polynomial coefficients: $C(Y) = c_n(x)Y^n + \cdots + c_1(x)Y + c_0(x) \in (\mathbb{C}[x])[Y]$. For example, the square root function $h(x) = \sqrt{x}$ is algebraic. Notice $(\sqrt{x})^2 - x = x - x = 0$. In particular, we have $h(x) = \sqrt{x}$ is a root of the non-zero polynomial $C(Y) = Y^2 - x \in (\mathbb{C}[x])[Y]$.

Building on our previous discussion about how collections of algebraic elements are closed under algebraic operations such as addition, subtraction, multiplication, division, and root taking, we should not be surprised to learn that functions such as $h(x) = \sqrt[5]{\dfrac{x^3 - \sqrt{x}}{\sqrt[3]{x^2+1} - 11}}$ are algebraic.[9]

4.1 Transcendental Functions

The main goal of this paper is to demonstrate that many familiar functions are transcendental. Interestingly, even though proving numbers such as π and e are transcendental (over $\mathbb{Q}$) is rather intricate and challenging (see [3]), proving trigonometric, exponential, and logarithmic functions are transcendental is rather elementary. We rely on the following simple theorem:

[8]Since we aim to study the algebraic versus the transcendental, we sweep issues of domain and analysis under the rug. Considering function domains, branch cuts, and polynomial functions versus formal polynomials would take us too far afield. The careful reader should be comforted to know that (when working in characteristic 0) there is algebraically essentially no difference between polynomial functions (treating x as a variable) versus the formal polynomials (treating x as an indeterminate).

[9]See §6 Reader Investigations #5.

Theorem 4.2 *Just like polynomial functions, the zero function is the only algebraic function with infinitely many zeros.*

Proof: Suppose $h(x)$ is algebraic. Therefore, there exists a non-zero polynomial $C(Y) \in (\mathbb{C}[x])[Y]$ such that $C(h(x)) = 0$, say $C(Y) = c_n(x)Y^n + \cdots + c_1(x)Y + c_0(x)$ where $c_n(x), \ldots, c_1(x), c_0(x) \in \mathbb{C}(x)$ and at least one $c_j(x) \neq 0$. Without loss of generality, we may assume $c_n(x) \neq 0$. Next, consider the greatest common divisor (GCD) of the coefficients of the powers of Y in $C(Y)$, say $d(x)$ is the GCD of the polynomials $c_n(x), \ldots, c_1(x), c_0(x)$. Then, $\frac{1}{d(x)}C(Y)$ is still a non-zero element of $(C[x])[Y]$ with root $h(x)$. Thus, without loss of generality, we also assume that the GCD of $c_n(x), \ldots, c_1(x), c_0(x)$ is 1.

Next, suppose that $h(x)$ has infinitely many zeros, say $z_1, z_2, \ldots$ are distinct complex numbers such that $h(z_1) = h(z_2) = \cdots = 0$. We can take our polynomial relation $0 = C(h(x)) = c_n(x)(h(x))^n + \cdots + c_1(x)h(x) + c_0(x)$ and evaluate it at any of these numbers. For any $k = 1, 2, \ldots$, we get $0 = C(h(z_k)) = c_n(z_k)(h(z_k))^n + \cdots + c_1(z_k)h(z_k) + c_0(z_k) = c_n(z_k) \cdot 0 + \cdots + c_1(z_k) \cdot 0 + c_0(z_k)$. Therefore, $c_0(z_k) = 0$ for all $k = 1, 2, \ldots$ so that $c_0(x)$ is a complex polynomial with infinitely many roots. Thus $c_0(x) = 0$ (i.e., $c_0(x)$ is the *zero polynomial*). We now have $C(Y) = c_n(x)Y^n + \cdots + c_1(x)Y + 0 = (c_n(x)Y^{n-1} + \cdots + c_1(x))Y$. Let $D(Y) = C(Y)/Y = c_n(x)Y^{n-1} + \cdots + c_1(x)$. Either $h(x)$ is a root of $D(Y)$ or $h(x)$ is a root of Y. The latter would mean $h(x) = 0$ (and so we are done). Otherwise, we have $h(x)$ is the root of a polynomial (in Y) of one degree less: $D(Y) = c_n(x)Y^{n-1} + \cdots + c_1(x)$. Since the GCD of $c_n(x), \ldots, c_1(x), c_0(x) = 0$ is 1, the GCD of $c_n(x), \ldots, c_1(x)$ is still 1. Thus, $D(Y)$ satisfies the same criterion as $C(Y)$ did but is of one degree lower.

Continuing in this fashion, we eventually must have that $h(x)$ is a root of $P(Y)Y$ where $P(Y)$ is a polynomial in $(\mathbb{C}[x])[Y]$ of degree zero whose coefficients have a GCD of 1. In other words, $h(x)$ is the root of $p_0(x)Y$ for some $p_0(x) \in \mathbb{C}[x]$ where the GCD $p_0(x)$ (by itself) is 1. Thus, $p_0(x)$ is a (non-zero) constant polynomial. Without loss of generality, we may assume $p(x) = 1$. Thus, $h(x)$ is a root of Y. Therefore, we cannot escape that $h(x)$ must be identically 0. ⧫

Armed with the above theorem, we can easily get that many familiar functions are transcendental functions.

Corollary 4.3 *Any non-constant periodic function is transcendental.*

Proof: Essentially any non-constant periodic function appropriately shifted by a constant yields a non-constant function with infinitely many zeros and thus must be transcendental by our theorem above.

In more detail, suppose $f(x)$ is non-constant and periodic with some period p. In other words, $f(z) = f(z+p) = f(z+2p) = \cdots = f(z+np)$ for any z in the domain of $f(x)$ and integer $n \in \mathbb{Z}$. Pick some z_0 in $f(x)$'s domain and define $c = f(z_0)$. Then the function $h(x) = f(x) - c$ has infinitely many zeros since given $z_k = z_0 + pk$ for $k \in \mathbb{Z}$, we have $h(z_k) = f(z_0 + pk) - c = f(z_0) - c = c - c = 0$. Also, since $f(x)$ is non-constant, $h(x) = f(x) - c \neq 0$. Thus, $h(x)$ is transcendental, and so $f(x)$ must be as well. ⧫

All of our familiar trigonometric and exponential functions are periodic – at least when working over the complex numbers.

Corollary 4.4 *The trigonometric functions:* $\sin(x)$, $\cos(x)$, $\tan(x)$, $\sec(x)$, $\csc(x)$, $\cot(x)$, *the hyperbolic trigonometric functions:* $\sinh(x)$, $\cosh(x)$, $\tanh(x)$, $\mathrm{sech}(x)$, $\mathrm{csch}(x)$, $\coth(x)$, *and the exponential function* e^x *are all transcendental.*

Proof: The trigonometric functions have periods 2π or π. The hyperbolic trigonometric functions and the exponential function have periods $2\pi i$ or πi. ♦

This also allows us to see that b^x is transcendental for any fixed base $b > 0$ (and $b \neq 1$).[10] Next, we could ask about inverse functions. Here, we have a very clear answer.

As a note, we have been fairly careless about considering domains of functions. Since the notion of being algebraic versus transcendental is about satisfying polynomial relations in an indeterminate, we should see that the specific domains of definition are not really that consequential. Along these lines, we refer to "an" inverse. Technically, we are referring to some branch of an otherwise multivalued inverse. This is in step with a typical Calculus sequence level treatment of functions. For example, when working over the real numbers, we refer to $\sqrt{x}$ as *the* inverse of x^2, but in reality we have two potential branches: $\sqrt{x}$ and $-\sqrt{x}$. Likewise, we refer to $\arcsin(x)$ as *the* inverse of the sine function, but this is just one of infinitely many possible branches of sine's "inverse". Working in the context of the complex numbers muddies these waters even more. Again, we sweep such issues under the rug and focus on algebraic relations. We will call $h^{-1}(x)$ an *inverse* of $h(x)$ if $h(h^{-1}(x)) = x$ for any x where the composition is defined.

Proposition 4.5 *An inverse of an algebraic function is algebraic. Therefore, an inverse of a transcendental function is transcendental.*

Proof: Let $h(x)$ be algebraic with inverse $h^{-1}(x)$ (i.e., $h(h^{-1}(x)) = x$). Since $h(x)$ is algebraic, we have $h(x)$ is the root of some non-zero polynomial $C(Y) = c_n(x)Y^n + \cdots + c_1(x)Y + c_0(x) \in (\mathbb{C}[x])[Y]$. We can expand each coefficient $c_j(x) = b_{m_j j}x^{m_j} + \cdots + b_{1j}x + b_{0j}$. Expanding out our polynomial and thinking about it as a polynomial in both Y and x, we have $B(x, Y) = \sum\limits_{i,j} b_{ij}x^iY^j$.

We have that $B(x, h(x)) = \sum_{i,j} b_{ij}x^i(h(x))^j = \sum_j b_j(x)(h(x))^j = 0$. We now make the substitution $x = h^{-1}(t)$ (using a different variable t for clarity) so that $h(x) = h(h^{-1}(t)) = t$. Thus, $0 = B(x, h(x)) = B(h^{-1}(t), t) = \sum_{i,j} b_{ij}(h^{-1}(t))^it^j$.

Briefly, we have $B(x, h(x)) = 0$ implies $B(h^{-1}(x), x) = 0$. In other words, $Y = h^{-1}(x)$ is a root of the non-zero polynomial $B(Y, x) = \sum\limits_{k,\ell} c_{\ell k}x^kY^\ell \in (\mathbb{C}[x])[Y]$. Thus $h^{-1}(x)$ is algebraic.

Next, if $h^{-1}(x)$ is algebraic, then $(h^{-1})^{-1}(x) = h(x)$ must be algebraic too. Therefore, either both $h(x)$ and $h^{-1}(x)$ are algebraic, or neither are (i.e., either both are algebraic or both are transcendental). ♦

To illustrate the above theorem, consider $\sqrt{x}$ is a root of $Y^2 - x$, and so its inverse x^2 should be a root of $x^2 - Y$ (which it is).

[10] Notice $b^x = e^{x\ln(b)} = e^{x\ln(b)+2\pi i} = e^{(x+2\pi i/\ln(b))\ln(b)} = b^{x+2\pi i/\ln(b)}$, so these (non-constant) functions have period $2\pi i/\ln(b)$.

Corollary 4.6 *Inverse trigonometric functions:* $\arcsin(x)$*,* $\arccos(x)$*,* $\arctan(x)$*,* $\text{arcsec}(x)$*,* $\text{arccsc}(x)$*,* $\text{arccot}(x)$*, inverse hyperbolic trigonometric functions:* $\text{arcsinh}(x)$*,* $\text{arccosh}(x)$*,* $\text{arctanh}(x)$*,* $\text{arcsech}(x)$*,* $\text{arccsch}(x)$*,* $\text{arccoth}(x)$*, and logarithm functions:* $\ln(x)$ *(as well as* $\log_b(x)$ *for positive base* $b \neq 1$*) are transcendental.*

5 Differential Algebra

Working with complex functions opens a door to consider yet another kind of closure. We might ask if we have closure under derivatives. Notice that the derivative of a polynomial (respectively a rational function) is still a polynomial (respectively a rational function). Thus both $\mathbb{C}[x]$ and $\mathbb{C}(x)$ are closed under differentiation. How about algebraic and transcendental functions?

Proposition 5.1 *The derivative of an algebraic function is an algebraic function.*

Proof: Let $h(x)$ be an algebraic function. In particular, suppose $h(x)$ is a root of $C(Y) = c_n(x)Y^n + \cdots + c_1(x)Y + c_0(x)$ where $c_n(x), \ldots, c_1(x), c_0(x) \in \mathbb{C}[x]$. Also, assume $C(Y)$ is a non-zero polynomial of lowest possible degree (i.e., it is a $\mathbb{C}[x]$-multiple of the minimal polynomial of $h(x)$).

We can use the product, power, and chain rules to help differentiate the relation $0 = C(h(x))$ and get

$$0 = c_n'(x)(h(x))^n + c_n(x)n(h(x))^{n-1}h'(x) + \cdots + c_1'(x)h(x) + c_1(x)h'(x) + c_0'(x).$$

Therefore, $-(c_n'(x)(h(x))^n + \cdots + c_1'(x)h(x) + c_0'(x)) = (nc_n(x)(h(x))^{n-1} + \cdots + c_1(x))h'(x)$. We note that $C(Y)$ had minimal degree such that $h(x)$ was a root, so $nc_n(x)(h(x))^{n-1} + \cdots + c_1(x)$ cannot be identically zero. Therefore, we can solve for $h'(x)$:

$$h'(x) = -\frac{c_n'(x)(h(x))^n + \cdots + c_1'(x)h(x) + c_0'(x)}{nc_n(x)(h(x))^{n-1} + \cdots + c_1(x)}.$$

Finally, notice that $h(x)$ and polynomials ($c_n'(x)$ and $nc_n(x)$ etc.) are algebraic. Also, algebraic functions form a field (i.e., closed under adding, subtracting, multiplying, and dividing by non-zero elements). Therefore, $h'(x)$ must be algebraic too. ♦

We note that the above proof not only shows the derivative of an algebraic function must be algebraic, but it also gives a formula for its derivative in terms of our algebraic function's minimal polynomial. For example, $h(x) = \sqrt{x}$ satisfies $Y^2 - x$. Thus, $h'(x) = -\frac{0(h(x))^2 - 1}{2h(x)} = \frac{1}{2\sqrt{x}}$ as we know from Calculus.

In contrast to algebraic functions, transcendental functions are not closed under differentiation. For example, we already know that $\arctan(x)$ is transcendental. However, its derivative, $\frac{1}{1+x^2}$, is algebraic. On the other hand, sometimes derivatives of transcendental functions do remain transcendental. For example, $\sin(x)$ is transcendental, as is its derivative $\cos(x)$.

From the results above and our experiences in Calculus, we are led to believe that differentiation leads us from more complicated functions to simpler ones. From an analytic perspective, this is definitely not the case![11] However, from a symbolic perspective, there is some truth. Differentiation moves us from more exotic to less exotic collections. Repeatedly differentiating polynomials eventually yields constants and then zero. Differentiating our inverse trigonometric and log functions yields rational functions or square roots of rational functions.

5.1 Elementary Functions

In Calculus, we want access to certain algebraic and exponential, logarithmic, and trigonometric functions. Functions built from these pieces are referred to as *elementary functions*. First, we note that trigonometric and inverse trigonometric functions are redundant when working in the world of complex variables. For example, $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$ and $\arcsin(x) = -i\ln(ix + \sqrt{1-x^2})$. Thus, up to some algebraic stuff, we can just consider exponentials and logarithms.

One more carefully defines an elementary function to be a function obtained by finitely many steps of exponentiating, taking a logarithm, and extracting a root of a polynomial. For example,

$$h(x) = \sqrt{\frac{e^{\sqrt[3]{x^5+1}} + x^{10} - 1}{\ln(x^3 + \sqrt{x^9 - 4}) + x^3 + \ln(e^{1/x} + 11)}}$$

is an elementary function.

One can show that while elementary functions are closed under our usual algebraic operations and differentiation, they are not closed under anti-differentiation (i.e., integration). For example, $\int e^{x^2}\,dx$ and $\int \frac{\sin(x)}{x}\,dx$ are not elementary functions [7]. A special part of the study of Differential Algebra known as Differential Galois Theory gives a more complete picture of how all of these things fit together. For example, Andy Magid [6] tells us that elementary functions are exactly those that live in an abelian differential field extension of $\mathbb{C}(x)$. Without explaining exactly what this means, we assure the reader that elementary functions (as non-elementary as they might seem) are a mathematically natural collection of functions for one to consider when working in a field of study such as Calculus.

6 Reader Investigations

The following is a set of investigatory exercises that use the computer algebra system Maple™. These problems can be easily modified for other computer algebra systems such as TI Nspire™, Sage, PocketCAS™, Mathematica™, or Wolfram Alpha™. A Maple™worksheet containing these investigations can be found at: https://BillCookMath.com/papers/Transcendental-Investigations.mw

1. We'll use Maple to explore some minimal polynomials for simple radicals. Start with:
   > *MP := PolynomialTools:-MinimalPolynomial*:

[11] Differentiation destroys. Integration smooths.

> *p3* := *MP*($\sqrt{3}, x$);

$$p3 := x^2 - 3$$

> *p5* := *MP*($\sqrt{5}, x$);

$$p5 := x^2 - 5$$

Now try:
> *p3p5* := *expand*(*p3·p5*);

$$p3p5 := x^4 - 8x^2 + 15$$

> *p15* := *MP*($\sqrt{15}, x$);

$$p15 := x^2 - 15$$

> *solve*(*p3p5*);

$$\sqrt{3}, -\sqrt{3}, \sqrt{5}, -\sqrt{5}$$

> *solve*(*p15*);

$$\sqrt{15}, -\sqrt{15}$$

Explain how combining the radicals affects the minimal polynomials and their roots.

Let's try:
> *MP*($\sqrt{3} + \sqrt{5}$*, x);*
solve(%);

$$x^4 - 16x^2 + 4$$

$$\sqrt{5} - \sqrt{3}, -\sqrt{5} + \sqrt{3}, \sqrt{3} + \sqrt{5}, -\sqrt{3} - \sqrt{5}$$

> *p3 + p5;*
solve(*%*);

$$2x^2 - 8$$

$$2, -2$$

What can we conclude about the minimal polynomials of combinations of radicals?

2. Let's consider a different combination of simple radicals. What does the following sequence of commands tell us?
> *MP := PolynomialTools:-MinimalPolynomial;*
> *r2 :=* $2^{1/2}$*;*
> *r8 :=* $8^{1/2}$*;*
> *minpolyr2 := MP(r2, x);*
> *minpolyr8 := MP(r8, x);*
> *roots2 := solve(minpolyr2);*
> *roots8 := solve(minpolyr8);*

3. Investigate minimal polynomials of constants and of simple polynomials using the interactive Maple document "Exploring Simple Minimal Polynomials" hosted on the Maple.cloud.

4. A minimal polynomial can be viewed as a level curve of a surface.
Execute the following in Maple.

> *with(plots):*
> *plotOpts := (style = surfacecontour, shading = XY, transparency = 0.5, thickness = 5):*
> *p2 := (x, Y)* $\rightarrow Y^2$ *- x;*
> *plot3d(p2(x, Y), x = -3 .. 3, Y = -3 .. 3, plotOpts, contours = [1]);*
> *display(%, orientation = [-90, 0, 0], axes = normal);*
What can we conclude from the graphs?

> *p3 := (x, Y)* $\rightarrow Y^3$ *- x;*
> *plot3d(p3(x, Y), x = -3 .. 3, Y = -3 .. 3, plotOpts, contours = [1]);*
> *display(%, orientation = [-90, 0, 0], axes = normal);*
What can we conclude from this pair of graphs?

Now interpret the result of:
> *p31 := (x, Y)* $\rightarrow 5 \cdot Y^3 - 3 \cdot Y \cdot x$*;*
> *plot3d(p31(x, Y), x = -3 .. 3, Y = -3 .. 3, plotOpts, contours = [1]);*
> *display(%, orientation = [-90, 0, 0], axes = normal);*
Identify the common feature from all of these graphs?

5. Do the following Maple commands verify that the function

$$h(x) = \sqrt[5]{\frac{x^3 - \sqrt{x}}{\sqrt[3]{x^2+1} - 11}}$$

given at the end of §4's introduction is algebraic or do they show that $h(x)$ is actually transcendental?
> *mp := evala@Minpoly:*
> $h := x \rightarrow ((x^3 - x^{1/2})/((x^2+1)^{1/3} - 11))^{1/5}$*;*
> *Ph := mp(h(x), Y);*
> *subs(Y = h(x), Ph);*
> *radsimp(%);*

6. Showing the negative result "There is no polynomial such that..." is very difficult. Execute:
> *mp := evala@Minpoly:*
> *Psin := mp(*$\sin(x)$*, Y);*

   (a) Is $f(x) = \sin(x)$ algebraic or transcendental? Explain your response.

   (b) What does this result tell us about the capabilities of Maple's *Minpoly* command?

7. Let's look at nesting roots and what happens to the minimal polynomials. Execute the following in Maple:
> *mp := evala@Minpoly:*

> *r1* := $\sqrt{2}$;
> **for** i **from** 2 **to** 4 **do** $r||i := sqrt(2 + r||(i-1))$ **end do**;
Now compute the minimal polynomials and then find the roots.
> **for** i **to** 4 **do** $p||i := mp(r||i, Y)$ **end do**;
> **for** i **to** 4 **do** $[solve(p||i = 0, Y)]$ **end do**;

(a) Do you see any patterns in the minimal polynomials?

(b) Do the polynomials share any common roots?

8. Proposition 4.5 concerns the inverse of an algebraic function. Define the function $g(x) = x^2 + \sqrt{x}$, then find its minimal polynomial.
> *mp := evala@Minpoly:*
> $f := x \to x^2 + \sqrt{x}$;
> *mp(f(x), Y);*
> *minpoly := sort(%, Y);*

Now verify that f is a root of the polynomial.
> *subs(Y = f(x), minpoly);*
> *expand(%);*

Compute an inverse for f.
> *solve(Y = f(x), x);*
> *finv := %;*

Proposition 4.5 says the inverse function f^{-1} satisfies the minimal polynomial of f with the variables interchanged ($Y \leftrightarrow x$).
> *subs(x = finv, minpoly);*
> *simplify(%);*
Did it work? Explain.

9. The (real-valued) Lambert-W function is a *special function* that is defined as the solution of the equation $ye^y = x$ for y. This function has import applications in quantum physics, biochemistry, and applied mathematics.
Define the function $f(x) = x \cdot e^x$ in Maple.
> $f := x \to x \cdot \exp(x)$;

Find the inverse function.
> *finv := solve(y = f(x), x);*

This inverse is analogous to $\arctan$ in that it has an infinite number of "branches".
Let's check it.
> *'f'(x) = f(x);*
> *LambertW(f(x)):*
> *simplify(%) assuming* $x > 0$:

> *%% = %;*

The other direction:
> *f(LambertW(y));*
> *simplify(%) assuming* $y > 0$*;*

To know more than you ever wanted, enter
> *FunctionAdvisor(LambertW);*

10. Proposition 5.1 states that if $f(x)$ is algebraic, then $f'(x)$ is also algebraic.

   (a) In Maple, define the absolute value $A(x) = |x|$ and the signum $S(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$ functions.

   (b) Plot the two functions together with

   $$> plot([A(x), S(x)], x = -2..2, -2..2, discont = [symbol = solidcircle]);$$

   How do the functions appear to be related from looking at the graph?

   (c) $A(x)$ is algebraic as it satisfies the polynomial $p(x, Y) = Y^2 - x^2$. However $S(x)$ is *not* algebraic! (An algebraic function is continuous on its domain.) Explain how this does not violate Proposition 5.1. Or does it? (Hint: Ask Maple to $convert($*diff*$(|x|, x),$ *piecewise*$)$.)

11. Examine the trigonometric functions written in terms of e^x and $\ln(x)$ for patterns by making charts in Maple. Which of the functions in the charts are *elementary functions*?

   (a) Trigonometric functions chart:
   $> T := \langle \cos(x), \sin(x), \tan(x), \sec(x), \csc(x), \cot(x) \rangle$:
   $> Te := map(convert, T, expln)$:
   $> \langle T \,|\, \langle \text{`}\leftrightarrow\text{`\$6} \rangle \,|\, Te \rangle$;

   (b) Inverse trigonometric functions chart:
   $> AT := \langle \arccos(x), \arctan(x), \arcsin(x), \text{arcsec}(x), \text{arccsc}(x), \text{arccot}(x) \rangle$:
   $> ATl := map(convert, AT, expln)$:
   $> \langle AT \,|\, \langle \text{`}\leftrightarrow\text{`\$6} \rangle \,|\, ATl \rangle$;

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